W38. Let $(a_n)_n$ be a sequence, given by the recurrence:

$$ma_{n+1} + (m-2)a_n - a_{n-1} = 0$$

where $m \in \mathbb{R}$ is a parameter and the first two terms of $(a_n)_n$ are fixed known real numbers. Find $m \in \mathbb{R}$, so that

$$\lim_{n\to\infty} a_n = 0$$

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Noting that for $m = 0$ the recurrence becomes $-2a_n - a_{n-1} = 0 \Leftrightarrow$
$a_n = -\frac{1}{2}a_{n-1}, n \in \mathbb{N} \Leftrightarrow a_n = \left(-\frac{1}{2}\right)^n a_0$ and, therefore, $\lim_{n\to\infty} a_n = 0$ in that case, we assume for further that $m \in \mathbb{R}\setminus\{0\}$.

Let p be solution of equation $p - \frac{1}{p} = m$, which is solvable for any $m \in \mathbb{R}\setminus\{0\}$, namely, $p := \frac{m + \sqrt{m^2+4}}{2}$.

Then $\left\{p \mid p = \frac{m+\sqrt{m^2+4}}{2} \text{ and } m \in \mathbb{R}\setminus\{0\}\right\} = \mathbb{R}\setminus\{0,1,-1\}$ and

$ma_{n+1} + (m-2)a_n - a_{n-1} = 0$ becomes, $\left(p - \frac{1}{p}\right)a_{n+1} + (p - \frac{1}{p} - 2)a_n - a_{n-1} = 0 \Leftrightarrow$

(1) $\quad (p^2-1)a_{n+1} + (p^2-2p-1)a_n - pa_{n-1} = 0, n \in \mathbb{N}.$

Since numbers $\frac{1}{p-1}, -\frac{p}{p+1}$ be solution of characteristic quadratic equation

$$(p^2-1)x^2 + (p^2-2p-1)x - p = 0$$

then $a_n = \frac{1}{(p-1)^n}\left(a_1 + \frac{pa_0}{p+1}\right) + \frac{(-1)^{n+1}p^n}{(p+1)^n}\left(a_1 - \frac{a_0}{p+1}\right), n \in \mathbb{N}\cup\{0\}.$

For any fixed a_0, a_1 the claim $\lim_{n\to\infty} a_n = 0$ is equivalent to $\begin{cases} \lim_{n\to\infty} \frac{1}{(p-1)^n} = 0 \\ \lim_{n\to\infty} \frac{p^n}{(p+1)^n} \end{cases} \Leftrightarrow$

$$\begin{cases} |p-1| > 1 \\ \left|\frac{p}{p+1}\right| < 1 \end{cases} \Leftrightarrow \begin{cases} \left[\begin{array}{l} p > 2 \\ p < 0 \end{array}\right. \\ -\frac{1}{2} < p \end{cases} \Leftrightarrow \left[\begin{array}{l} p > 2 \\ -\frac{1}{2} < p < 0 \end{array}\right. .$$

Thus, $\lim_{n\to\infty} a_n = 0$ iff $m = 0$ or $m = h(p) := p - \frac{1}{p},\ p \in (2,\infty) \cup \left(-\frac{1}{2}, 0\right)$.

Since $p - \frac{1}{p}$ increase on $(0,\infty)$ and on $(-\infty,0)$ then $h(2,\infty) = (3/2,\infty)$ and $h((-1/2,0)) = (3/2,\infty)$ and, therefore, $\lim_{n\to\infty} a_n = 0$ iff $m \in (3/2,\infty) \cup \{0\}$.